4. *If* $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, *prove that* $\frac{dy}{dx} = \frac{2}{1+x^2}$. (isc 2003)

Solution: Given $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$= \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$ Let $x = \tan\theta$

$= \tan^{-1}(\tan 2\theta)$ $\therefore\ \theta = \tan^{-1}x$

$= 2\theta$

$= 2\tan^{-1}x$

$\therefore\ \frac{dy}{dx} = \frac{2}{1+x^2}$ Proved.

5. *If* $y = e^{m\cos^{-1}x}$, *prove that* $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2 y$ (isc 2003)

Solution: Given $y = e^{m\cos^{-1}x}$

Taking log both sides

$\Rightarrow\ \log y = m\cos^{-1}x$

Differentiating each term w.r.t. x

$\Rightarrow\ \frac{1}{y}\frac{dy}{dx} = -\frac{m}{\sqrt{1-x^2}}$

$\Rightarrow\ \sqrt{1-x^2}\frac{dy}{dx} = -m\,y$

Squaring both sides

$\Rightarrow\ (1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2 y^2$

Differentiating each term w.r.t. x

$\Rightarrow\ (1-x^2)2\frac{dy}{dx}\frac{d^2y}{dx^2} + (-2x)\left(\frac{dy}{dx}\right)^2 = m^2\,2y\frac{dy}{dx}$

Dividing each term by $2\frac{dy}{dx}$

$\Rightarrow\ (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2 y$ Proved

6. *If* $x^y y^x = 5$, *show that* $\frac{dy}{dx} = -\left(\frac{\log y + \frac{y}{x}}{\log x + \frac{x}{y}}\right)$ (isc 2004)

Solution: Given $x^y y^x = 5$

Taking log both sides

$\Rightarrow\ \log x^y + \log y^x = \log 5$

$\Rightarrow\ y\log x + x\log y = \log 5$

Differentiating each term w.r.t. x

$\Rightarrow\ y\frac{1}{x} + \log x\frac{dy}{dx} + \log y + x\frac{1}{y}\frac{dy}{dx} = 0$

$\Rightarrow\ \left(\log x + \frac{x}{y}\right)\frac{dy}{dx} = -\frac{y}{x} - \log y$

$\Rightarrow\ \frac{dy}{dx} = -\left(\frac{\log y + \frac{y}{x}}{\log x + \frac{x}{y}}\right)$ Proved

7. *If* $x = a\sin^3 t$ *and* $y = a\cos^3 t$, *find* $\frac{dy}{dx}$. (isc 2004)

Solution: Given $x = a\sin^3 t$ | $y = a\cos^3 t$

Differentiating $w.r.t.\ t$

$\Rightarrow\ \frac{dx}{dt} = a.3\sin^2 t.\cos t$ | $\Rightarrow\ \frac{dy}{dt} = a\,3\cos^2 t.(-\sin t)$