$$\Rightarrow \quad \frac{dy}{dx} = \cos(x^2).2x \qquad and \ \frac{dz}{dx} = 3\,x^2$$

$$\therefore \ \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{2\,x\,\cos x^2}{3\,x^2}$$

$$= \frac{2\,\cos x^2}{3\,x} \qquad \text{Answer.}$$

18. Using a suitable substitution, find the derivative of $\tan^{-1}\sqrt{\frac{a-x}{a+x}}$ with respect to x.

$(isc\ 2010)$

Solution: $Let \quad y = \tan^{-1}\sqrt{\frac{a-x}{a+x}}$ $\qquad Let\ x = a\ \cos\theta$

$$= \tan^{-1}\sqrt{\frac{a-a\ \cos\theta}{a+a\ \cos\theta}} \qquad \therefore\ \theta = \cos^{-1}\left(\frac{x}{a}\right)$$

$$= \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \tan^{-1}\sqrt{tan^2\left(\frac{\theta}{2}\right)}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

$$\therefore\ \frac{dy}{dx} = \frac{1}{2}\cdot\frac{-1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\frac{1}{a}$$

$$= -\frac{1}{2\sqrt{a^2-x^2}} \qquad \text{Answer}$$

19. $If\ \ y = x^x,\ \ prove\ that\ \ \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$ $\qquad (isc\ 2011)$

Solution : Given $\quad y = x^x$

Taking log both sides

$$\Rightarrow \quad \log y = x\ \log x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \log x \qquad \text{(differentiating each term } w.r.t.\ x)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

$$\Rightarrow \quad \frac{1}{y}\frac{d^2y}{dx^2} + \frac{dy}{dx}.\left(-\frac{1}{y^2}\right)\frac{dy}{dx} = \frac{1}{x} \qquad \text{(differentiating again } w.r.t.\,x)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0 \qquad \text{Proved.}$$