20. $If \quad e^y(x+1) = 1, then\ show\ that\ \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2.$ $(isc\ 2012)$

Solution: Given $e^y(x+1) = 1$

$\Rightarrow \quad e^y = \frac{1}{x+1}$

$\Rightarrow \quad y = \log\left(\frac{1}{x+1}\right)$

$\Rightarrow \quad y = -\log(x+1)$

$\therefore \quad \frac{dy}{dx} = -\frac{1}{x+1}$ and $\left(\frac{dy}{dx}\right)^2 = \left(\frac{1}{x+1}\right)^2$ ---------- (1)

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{(-1)}{(x+1)^2}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{1}{x+1}\right)^2$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$ [comparing with equation (1)] Proved

21. $If \quad y = (\cot^{-1}x)^2, show\ that\ (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$ $(isc\ 2013)$

Solution: Given $y = (\cot^{-1}x)^2$

Differentiating $w.r.t.\ x$

$\Rightarrow \quad \frac{dy}{dx} = 2\cot^{-1}x.\left(\frac{-1}{1+x^2}\right)$

$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = -2\cot^{-1}x$

$Differentiating\ again\ w.r.t.\ x$

$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$

$\Rightarrow \quad (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2$ Proved.

22. $If \quad y = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}, prove\ that\ (1-x^2)\frac{dy}{dx} = x + \frac{y}{x}$ $(isc\ 2014)$

Solution: Given $y = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$ ---------- (1)

$\Rightarrow \quad \sqrt{1-x^2}.y = x\sin^{-1}x$

$Differentiating\ w.r.t.\ x$

$\Rightarrow \quad \sqrt{1-x^2}\frac{dy}{dx} + \frac{1}{2}.\frac{1}{\sqrt{1-x^2}}.(-2x)y = x.\frac{1}{\sqrt{1-x^2}} + \sin^{-1}x$

$multiplying\ each\ term\ by\ \sqrt{1-x^2}$

$\Rightarrow \quad (1-x^2)\frac{dy}{dx} - xy = x + \sin^{-1}x\ .\sqrt{1-x^2}$

[putting value of $\sin^{-1}x$ from (1)]

$\Rightarrow \quad (1-x^2)\frac{dy}{dx} - xy = x + \frac{y\sqrt{1-x^2}}{x}.\sqrt{1-x^2}$

$\Rightarrow \quad (1-x^2)\frac{dy}{dx} - xy = x + \frac{y(1-x^2)}{x}$

$\Rightarrow \quad (1-x^2)\frac{dy}{dx} - xy = x + \frac{y}{x} - yx$

$\Rightarrow \quad (1-x^2)\frac{dy}{dx} = x + \frac{y}{x}$ Proved.

23. $If \quad y = e^{m\cos^{-1}x}, prove\ that\ (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = m^2y$ $(isc\ 2015)$

Solution: Given $y = e^{m\cos^{-1}x}$