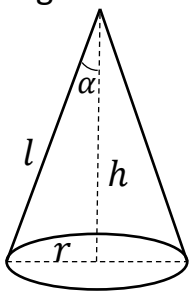
4. Prove that the semi-vertical angle of the right circular cone of given volume and least curved area is $\cot^{-1}\sqrt{2}$ (ISC 2023)

Solution: Let r, h, l be the radius, height, and slant height of the right circular cone respectively. So $l^2 = h^2 + r^2$.



Let V, S be the volume and curved surface area of the cone. Volume $V = \frac{1}{3}\pi r^2 h \Rightarrow h = \frac{3V}{\pi r^2}$ --- (i)

And $S = \pi r l \Rightarrow S^2 = \pi^2 r^2 l^2$

Let $A = S^2 = \pi^2 r^2 l^2$ ---------- (ii)

If S is least, S^2 $i.e.$, A will be least.

$\therefore A = \pi^2 r^2 (h^2 + r^2) = \pi^2 r^2 \left(\frac{9V^2}{\pi^2 r^4} + r^2\right) = \frac{9V^2}{r^2} + \pi^2 r^4$

$\therefore \frac{dA}{dr} = 9V^2 \times \frac{-2}{r^3} + 4\pi^2 r^3 = -\frac{18V^2}{r^3} + 4\pi^2 r^3$

And $\frac{d^2A}{dr^2} = -18V^2 \times \frac{-3}{r^4} + 4\pi^2 \times 3r^3 = \frac{54V^2}{r^4} + 12\pi^2 r^2 > 0$

$\therefore$ for least curved area, $\frac{dA}{dr} = 0$

$$\Rightarrow -\frac{18V^2}{r^3} + 4\pi^2 r^3 = 0$$

$$\Rightarrow \frac{18V^2}{r^3} = 4\pi^2 r^3$$

$$\Rightarrow 9\left(\frac{1}{3}\pi r^2 h\right)^2 = 2\pi^2 r^6$$

$$\Rightarrow \pi^2 r^4 h^2 = 2\pi^2 r^6$$

$$\Rightarrow h^2 = 2r^2 \Rightarrow h = \sqrt{2}\, r \Rightarrow \frac{h}{r} = \sqrt{2}$$

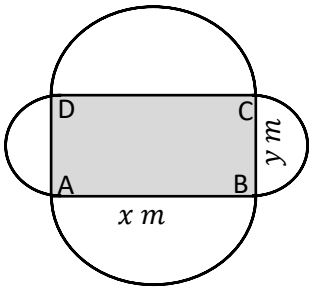
Let α be the semi-vertical angle of the cone.

$$\therefore \cot\alpha = \frac{h}{r} = \sqrt{2} \Rightarrow \alpha = \cot^{-1}\sqrt{2}$$

Hence the semi-vertical angle of the cone is $\cot^{-1}\sqrt{2}$ for least curved area of the right circular cone. Proved

5. A running track of 440 m is to be laid out enclosing a football field. The football field is in the shape of a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the length of its sides. Also calculate the area of the football field. ISC 2023

Solution: Let ABCD be the rectangular football field with a semicircle at each end. Let length and breadth of the rectangular field be $x\ m$ and $y\ m$ respectively.



Length of the running track = 440 m

$\therefore \pi \times \frac{x}{2} + \pi \times \frac{y}{2} + \pi \times \frac{x}{2} + \pi \times \frac{y}{2} = 440$

$\Rightarrow \pi x + \pi y = 440$

$\Rightarrow \pi(x + y) = 440$